1 Consumer and Producer Surplus

Every time you go to the supermarket and purchase something, you benefit (or at least you expect to benefit), otherwise you wouldn’t have made the purchase. Similarly, the owner of the supermarket benefits, otherwise they wouldn’t have sold the item too you. Measuring how much each of you benefits is a tricky question, especially if we want to compare them. In economics, we solve the problem by not actually trying to measure the benefits; rather, we try to measure the dollar value of the benefits to both you and the supermarket owner. Then we can add these up and use it as a sort of crude approximation to the total benefits to everyone in the market (even though it’s not clear what exactly we mean by “total benefits” but we’ll leave that discussion for philosophy class).

When we measure the dollar value of all benefits in a market, we split it into two parts — the supply side and the demand side. Consumer surplus measures the demand side — it is the difference between the amount of money the consumer is willing to pay for a good and the amount of money they did pay, summed across each unit of the good purchased (which may be worth less and less as more are purchased) and summed across all consumers. Producer surplus measures the supply side — it is the difference between the amount of money the producer receives for the good and the amount they were willing to sell the good for, summed over each good sold and each producer.

Suppose we have the following supply and demand curves:

\[ Q^D = 105 - 5P \]
\[ Q^S = -5 + 15P \]
The equilibrium price and quantity is \( P^* = 5.5 \) and \( Q^* = 77.5 \). Visually, we can see the consumer and producer surplus below. The consumer surplus is shaded red and the producer surplus is shaded blue. The area of the red area is the consumer surplus and the area of the blue area is the producer surplus. We can compute these areas using our favourite formula from middle school (or maybe grade school) for the area of a right triangle:

\[
\text{Area} = \text{base} \times \text{height}
\]

In both cases, the base is just equal to the equilibrium quantity — it’s the distance from the equilibrium point to the price axis. The height, on the other hand, is the distance between the equilibrium price and the relevant choke price. The supply choke price is the price at which quantity demanded is zero. We can find this by plugging \( Q^S = 0 \) into the supply curve, then solving for \( P \) to get \( P^S_{\text{choke}} = \frac{1}{3} \). The demand choke price is the price at which quantity supplied is zero. We can find this by plugging \( Q^D = 0 \) into the demand curve, then solving for \( P \) to get \( P^D_{\text{choke}} = 21 \).

Then we can use these numbers to find the consumer and producer surpluses:

\[
\text{CS} = \frac{1}{2} \times 77.5 \times (21 - 5.5) = 600.625
\]
\[
\text{PS} = \frac{1}{2} \times 77.5 \times (5.5 - \frac{1}{3}) = 200.2083
\]

So in this market, consumers receive $600.63 worth of benefits, and producers receive $200.21 worth of benefits. The total welfare (or surplus) in the market is the sum of these two, $800.84. When evaluating some public policy, economists usually use the difference between current total welfare and total welfare under the new policy as a measure of how much good that policy will do. This implicitly assumes that it doesn’t matter who gets the benefits and it also assumes that satisfying any person’s preference is a benefit. For example, giving a heroine addict heroine counts as a benefit. If the addict causing his family pain or robs a convenience store because he took heroine, that counts as a cost since the people he did that to would prefer if he hadn’t.

2 Calculus of Consumer and Producer Surplus

Recall from calculus that a definite integral is just the area under a curve. For example:

\[
\int_0^1 x \, dx
\]

is just the area of the shaded region below:
So now we can use calculus in order to determine the producer and consumer surplus.

2.1 Consumer Surplus

Let’s start with the demand curve. Consumer surplus is the difference between the price the consumer is willing to pay and the price they actually paid. The price that was actually paid is the equilibrium price, \( P^* \). The price that the consumer was willing to pay is the point on the demand curve corresponding to a particular quantity. For example, some consumers might be willing to pay $10 for the good, corresponding to \( Q^D = 55 \) for our demand curve, while others might be willing to pay $20, corresponding to \( Q^D = 5 \). The consumer surplus for each of these two consumers is $10 - $5.5 = $4.5 and $20 - $5.5 = $14.5 respectively. Now we just need to add all of these consumer surpluses up at each point along the demand curve to get total consumer surplus. Since there are infinitely many points on the demand curve, we are integrating, which just is a strange sort of summation. The integration is with respect to \( Q^D \) since each \( Q^D \) picks out a unique point on the demand curve.

To integrate with respect to \( Q^D \) we need the inverse demand curve, i.e. price as a function of quantity demanded. To get this we take the demand curve and solve for \( P \):

\[
Q^D = 105 - 5P \\
5P = 105 - Q^D \\
P = 21 - \frac{1}{5}Q^D.
\]

So the inverse demand curve is the function

\[
P(Q^D) = 21 - \frac{1}{5}Q^D.
\]

Now to compute consumer surplus, we need to integrate under the inverse demand curve but above the equilibrium price line, from \( Q^D = 0 \) to \( Q^D = Q^* \), the equilibrium quantity. In other words

\[
CS = \int_0^{Q^*} [P(Q^D) - P^*] \, dQ^D = \int_0^{77.5} \left[ 21 - \frac{1}{5}Q^D - 5.5 \right] \, dQ^D \\
= \left[ 15.5Q^D - \frac{1}{10}(Q^D)^2 \right]_0^{77.5} = 15.5(77.5) - \frac{1}{10}77.5^2 - \left[ 15.5(0) - \frac{5}{2}0^2 \right] = 600.625.
\]
This gives us the same answer as the triangle method, but has an added advantage — we can use it for demand curves that aren’t a straight line.

2.2 Producer Surplus

Figuring out producer surplus is almost exactly the same, except now the surplus is defined as the equilibrium price minus the price on the supply curve, integrated over quantity supplied, $Q^S$. In order to do this, we now need the inverse supply curve. We obtain the inverse supply curve analogously to how we obtained the inverse demand curve: by solve the supply curve for price.

$$Q^S = -5 + 15P$$

$$15P = 5 + Q^S$$

$$P = \frac{1}{3} + \frac{Q^S}{15}.$$

So inverse supply is

$$P(Q^S) = \frac{1}{3} + \frac{Q^S}{15}.$$

Then to compute producer surplus we just need to integrate the difference between the equilibrium price and inverse supply over quantity supplied, from zero to the equilibrium quantity:

$$PS = \int_0^{Q^*} [P^* - P(Q^S)] \, dQ^S = \int_0^{77.5} \left[ 5.5 - \frac{1}{3} - \frac{Q^S}{15} \right] \, dQ^S$$

$$= \left[ \frac{31}{6} Q^S - \frac{1}{30} (Q^S)^2 \right]_0^{77.5} = \frac{31}{6} \cdot 77.5 - \frac{1}{30} \cdot 77.5^2 - \left[ \frac{31}{6} (0) - \frac{1}{30} (0) \right] = 200.2083.$$

This is again the same as before using the area formula.

2.3 Nonlinear Supply and Demand Curves

The real power of the calculus method comes from making our supply and demand curves more realistic — i.e. by making them actual curves rather than straight lines. For example, suppose our inverse supply and demand curves are

$$P = 50 + .01(Q^S)^2$$

$$P = 250 - .03(Q^D)^2$$

The equilibrium quantity and price and $Q^* = \sqrt{5000} = 70.71$ and $P^* = 100$. We can see the curves plotted below with the consumer and producer surpluses shaded in red and blue respectively.
Now in order to compute consumer and producer surplus, we do exactly the same thing as before:

\[
CS = \int_0^{Q^*} [P(Q^D) - P^*] dQ = \int_0^{70.71} [250 - 0.03Q^2 - 100] dQ
\]

\[
= \left[150Q - 0.01Q^3\right]_0^{70.71} = 7071.068
\]

\[
PS = \int_0^{Q^*} [P^* - P(Q^S)] dQ = \int_0^{70.71} [100 - 50 - 0.01Q^2] dQ
\]

\[
= \left[50Q - \frac{0.01}{3}Q^3\right]_0^{70.71} = 2357.023
\]

3 Price Floors and Ceilings

Under a price floor or a price ceiling calculating consumer surplus and producer surplus is basically the same once we figure out the quantity actually sold on the market under the price regulation.

3.1 Price Floor

Suppose we have a price floor larger than the equilibrium price, \( P_f > P^* \) (if the price floor is lower than the equilibrium price it has no effect and usual equilibrium holds). Let \( Q_f = Q^D(P_f) \), i.e. the quantity demanded at that price. Then the formulas for consumer and producer surplus are

\[
CS_f = \int_0^{Q_f} [P(Q^D) - P_f] dQ
\]

\[
PS_f = \int_0^{Q_f} [P_f - P(Q^S)] dQ.
\]

In other words, the formulas are the same as the usual formulas for consumer and producer surplus, except we replace the equilibrium price and quantity with the floor price and the quantity demanded at that floor price. We use the quantity demanded because it will be less than the quantity supplied — producers may produce more than the equilibrium price, but those produced goods aren’t sold because the higher price scares of consumers.
Deadweight loss represents the possible benefits to either consumers or producers that could have been obtained in an open market that aren’t obtained because of the regulation. There are two methods to compute deadweight loss. Let \( CS_e \) and \( PS_e \) denote the unregulated consumer and producer surpluses, respectively, and \( Q^* \) denote the equilibrium quantity. Then deadweight loss can be computed as

\[
DWL = CS_e + PS_e - CS_f - PS_f
\]

\[
DWL = \int_{Q_f}^{Q^*} [P(Q^D) - P(Q^S)] \, dQ.
\]

The second formula integrates the difference between the inverse demand and inverse supply curves from the quantity sold at the price floor to the quantity sold and the equilibrium price.

### 3.2 Price Ceiling

Suppose we have a price ceiling smaller than the equilibrium price, \( P_c < P^* \) (if the price ceiling is larger than the equilibrium price it has no effect and usual equilibrium holds). Let \( Q_c = Q^S(P_c) \), i.e. the quantity sold at that price. Then the formulas for consumer and producer surplus are

\[
CS_c = \int_{0}^{Q_c} [P(Q^D) - P_c] \, dQ
\]

\[
PS_c = \int_{0}^{Q_c} [P_c - P(Q^S)] \, dQ.
\]

Once again the formulas are almost exactly the same. Here we use the quantity supplied as the upper bound of the integral because it will be less than the quantity demanded — consumer may want to buy more than the equilibrium quantity at this price, but they can’t buy more than the producers are willing to sell, and this will be less than the equilibrium quantity.

We can once again compute deadweight loss in two ways:

\[
DWL = CS_e + PS_e - CS_c - PS_c
\]

\[
DWL = \int_{Q_c}^{Q^*} [P(Q^D) - P(Q^S)] \, dQ.
\]

The only change here is that the lower bound of the integral formula is the new quantity sold on the market.

### 4 Guidelines for CS, PS, and DWL

Each one of these quantities has to be an integral of some sort. Consumer surplus is:

\[
CS = \int_{0}^{\tilde{Q}} [P(Q^D) - \tilde{P}] \, dQ
\]

where \( \tilde{Q} \) is the actual quantity sold on the market (i.e. the equilibrium quantity if there are no regulations), \( P(Q^D) \) is the inverse demand curve, and \( \tilde{P} \) is the actual price on the market (i.e. the equilibrium price if there are no regulations).

Producer surplus is:

\[
PS = \int_{0}^{\tilde{Q}} [\tilde{P} - P(Q^S)] \, dQ
\]

where again \( \tilde{Q} \) is the actual quantity sold on the market, \( \tilde{P} \) is the actual price in the market, and \( P(Q^S) \) is the inverse supply curve.
Deadweight loss is:

$$DWL = CS_e + PS_e - CS - PS$$

where $CS_e$ and $PS_e$ are what the consumer and producer surpluses would be if there were no regulations while $CS$ and $PS$ are what the actual consumer and producer surpluses are. Alternatively we can use the calculus formula:

$$DWL = \int_{Q}^{Q^*} [P(Q^D) - P(Q^S)] \, dQ$$

where $Q^*$ is the equilibrium quantity, $\hat{Q}$ is the actual quantity sold on the market, $P(Q^D)$ is the inverse demand curve and $P(Q^S)$ is the inverse supply curve.

Using this, you should be able to, for example, compute the CS, PS and DWL that results from a tax — just take care to distinguish the price the consumer pays for the good (which includes the tax) and the price the producer receives for the good (which does not include the tax since the government gets that).

5 CS, PS, and DWL under a tax

Suppose that the government institutes a tax of $T on a the sale of a some good. This decreases the price that producers actually receive by the value of the tax, which shifts the supply curve up by exactly $T. Mathematically, this means we have a new inverse supply curve:

$$P_T(Q^S) = P(Q^S) + T.$$ 

The inverse supply curve tells says: “you give me a quantity and I’ll tell you what the price has to be in order to convince the producers to produce that much.” So if the government imposes a tax on producers of $T for each unit of the good produced, the producers will effectively only be receiving $P - T$ for each unit they sell where $P$ is the sticker price. So in order to convince producers to produce and sell $Q^S$ units of the good, they must now receive $P_T(Q^S) = P(Q^S) + T$ per unit.

To compute the equilibrium quantity under the tax, $Q_T$, we use the new inverse supply, $P_T(Q^S)$, and the original inverse demand, $P(Q^D)$, set them equal and solve for $Q$. Next we plug $Q_T$ back into the inverse demand curve or the original inverse supply curve in order to obtain $P_T^D$, the price the consumers pay for the good. Then price that producers actually receive is $P_T^S = P_T^D - T$ since the value of the tax goes to the government. Then we can compute the government revenue, which is just the tax times the quantity sold. We can also compute consumer and producer surpluses, which are basically the same as before except now the two integrals use different prices instead of the same price. In other words (where GR is government revenue):

$$GR_T = Q_T \times T$$
$$CS_T = \int_{0}^{Q_T} [P(Q^D) - P_T^D] \, dQ$$
$$PS_T = \int_{0}^{Q_T} [P_T^S - P(Q^S)] \, dQ$$
$$DWL_T = \int_{Q}^{Q^*} [P(Q^D) - P(Q^S)] \, dQ = CS^* + PS^* - CS_T - PS_T - GR_T.$$ 

The second formula uses the untaxed market equilibrium consumer and producer surplus, $CS^*$ and $PS^*$ and it easier than to use than the integral if you already have $CS^*$ and $PS^*$. The graph below shows the consumer and producer surplus, deadweight loss, and government revenue.
If the tax is imposed on the consumer instead of the producer, nothing changes! Here the inverse demand curve is shifted by the tax instead of the inverse supply curve, so \( P_T(Q^D) = P(Q^D) - T \). This just reflects that if the consumer has to pay a tax of $T every time they purchase the good, the real price they pay is \( P + T \), so in order to induce the consumer to purchase \( Q^D \) units of the good with the tax in place, the price must be \( P(Q^D) - T \).

So we do the same thing to solve for \( Q_T \), the equilibrium quantity under the tax on consumers: set \( P(Q^S) = P_T(Q^D) \) and solve for \( Q \). Note how similar this is to the tax on producer case:

\[
\begin{align*}
\text{set } & P(Q^S) = P_T(Q^D) = P(Q^D) - T \text{ tax on consumers} \\
\text{set } & P(Q^D) = P_T(Q^S) = P(Q^S) + T \text{ tax on consumers}.
\end{align*}
\]

These two equations are equivalent! So solving each equation will give the same \( Q_T \), and so the price the consumers pay, the price the producers receive, consumer surplus, producer surplus, government revenue, and deadweight loss all must be the same! The graph below illustrates this.
A per unit subsidy is just a negative tax, so we can use the same math to analyze this case with no changes. The graph, however, is pretty convoluted because consumer surplus, producer surplus, and the cost to the government (i.e. negative government revenue) overlap so I don’t give one here. See page 101 in the textbook for an example. Also see the solutions to chapter 3 in class exercise 2 (posted on the course website) for examples of how to compute consumer surplus, producer surplus, government revenue, and deadweight loss for a tax and a subsidy.