Answer all questions on a separate sheet of paper. Make sure you show your work.

1. Question 3 on page 163 of the textbook, all parts.

Solution:
We have $P_X = 2$ and $P_Y = 1$ with two consumers A and B with their own utility functions and income:

- $U_A(X, Y) = X^{0.5}Y^{0.5}$, $I_A = 100$
- $U_B(X, Y) = X^{0.8}Y^{0.2}$, $I_B = 100$

These utility functions are both examples of a Cobb-Douglas utility function which pops up over and over again. Instead of solving the two consumer’s problems separately, we’ll solve the general problem for a consumer with a Cobb-Douglas utility function and plug in the results (note: this is also on page 159 in your textbook). The Cobb-Douglas utility function is

$$U(X, Y) = X^\alpha Y^{1-\alpha}$$

where $0 < \alpha < 1$. Now we can set up the Lagrangian:

$$L(X, Y, \lambda) = U(X, Y) + \lambda(I - P_X X - P_Y Y) = X^\alpha Y^{1-\alpha} + \lambda(I - P_X X - P_Y Y)$$

The first order conditions are:

$$\frac{\partial L}{\partial X} = 0$$
$$\frac{\partial L}{\partial Y} = 0$$
$$\frac{\partial L}{\partial \lambda} = 0$$

which in terms of our problem translates into

$$\alpha X^{\alpha-1}Y^{1-\alpha} - \lambda P_X = 0$$
$$\lambda(I - \alpha X^\alpha Y^{-\alpha} - \lambda P_Y = 0$$
$$I - P_X X - P_Y Y = 0.$$ 

We’ll rearrange these a little bit:

$$\alpha X^{\alpha-1}Y^{1-\alpha} = \lambda P_X$$
$$\lambda(I - \alpha X^\alpha Y^{-\alpha} = \lambda P_Y$$
$$P_X X + P_Y Y = I.$$ 

Now we want to solve for $X$ and $Y$. First, we’ll divide the first equation by the second to get:

$$\frac{\alpha X^{\alpha-1}Y^{1-\alpha}}{(1 - \alpha)X^\alpha Y^{-\alpha}} = \frac{\lambda P_X}{\lambda P_Y}$$

We can simplify the left and right sides a little bit to obtain:

$$\frac{\alpha Y^{1-\alpha}}{1 - \alpha} \frac{X^\alpha X^{-\alpha}}{P_X P_Y} = \frac{P_X}{P_Y}$$
And left hand side simplifies a little bit more:
\[ \frac{\alpha}{1-\alpha} \frac{Y}{X} = \frac{P_X}{P_Y} \]

And now we can solve for \( Y \):
\[ Y = \frac{1-\alpha}{\alpha} \frac{P_X}{P_Y} \]

Now we plug this back into the budget constraint:
\[ I = P_X X + P_Y \frac{1-\alpha}{\alpha} \frac{P_X}{P_Y} X = P_X X + \frac{1-\alpha}{\alpha} P_X X \]
\[ = \frac{P_X}{\alpha} X \]

so that
\[ X = \alpha \frac{I}{P_X}. \]

Then
\[ Y = \frac{1-\alpha}{\alpha} \frac{P_X}{P_Y} \alpha \frac{I}{P_X} = (1-\alpha) \frac{I}{P_Y} \]

so that the optimal quantities consumed of good \( X \) and good \( Y \) are:
\[ X = \alpha \frac{I}{P_X}. \]
\[ Y = (1-\alpha) \frac{I}{P_Y}. \]

Now to applied these formulas to our problem, we have:
\[ X_A = 0.5 \frac{100}{2} = 25 Y_A = 0.5 \frac{100}{1} = 50 \]

and
\[ X_A = 0.8 \frac{300}{2} = 120 Y_A = 0.2 \frac{300}{1} = 60 \]

In order to answer parts b) and c), there’s an easy trick you can use: the marginal rate of substitution is always equal to the price ratio. Here’s why. Recall that the marginal rate of substitution between goods \( X \) and \( Y \) for a given consumer is:
\[ MRS_{XY} = \frac{\partial U}{\partial X} \frac{\partial X}{\partial Y} \]

i.e. the ratio of derivatives. Now recall from the consumer’s problem the optimal solution requires
\[ \frac{\partial L}{\partial X} = 0 \]
\[ \frac{\partial L}{\partial Y} = 0 \]

But since \( L(X,Y,\lambda) = U(X,Y) + \lambda(I - P_X X - P_Y Y) \) we can rewrite this as
\[ \frac{\partial U}{\partial X} = \lambda P_X \]
\[ \frac{\partial U}{\partial Y} = \lambda P_Y. \]

Now if we divide the top equation by the bottom equation we have:
\[ \frac{\frac{\partial U}{\partial X}}{\frac{\partial U}{\partial Y}} = \frac{P_X}{P_Y} \]

i.e.
\[ MRS_{XY} = \frac{P_X}{P_Y}. \]

So it doesn’t matter who the consumer is - i.e. what their preferences are or what their income it - the marginal rate of substitution between any two goods for the consumer is always equal to the price ratio — it’s the same for everyone! So \( MRS_{XY} = 2/1 = 2 \) for both consumer A and consumer B in part b, and also for consumer C in part c.